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GUARANTEEING LANEKEEPING PERFORMANCE WITH TIRE SATURATION USING COMPUTED POLYNOMIAL LYAPUNOV FUNCTIONS

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ABSTRACT

Lanekeeping assistance systems hold the promise to save thousands of lives every year by preventing unintended lane departure. The potential field lanekeeping assistance system assists the driver in the lanekeeping task by effectively placing the vehicle in an artificial potential well with minimum at lane center. Previous work mathematically guarantees the performance of the system in the linear region of tire forces, but no guarantees of performance or even stability exist for saturating tires. These guarantees are crucial to ensure safety when the vehicle speed is too high for a given turn or the friction coefficient of the road is low due to surface conditions. Here we explore ways to numerically find Lyapunov functions for a vehicle with lanekeeping assistance and realistic tires. First the nonlinearity is modeled as a sector bounded disturbance, and a Lyapunov function is found for all vehicle trajectories that fit this sector bounded disturbance. Next, a polynomial fit is performed on the HSRI tire model, and a Lyapunov function is found for these polynomial dynamics. Each of these approaches provide Lyapunov functions valid well into the nonlinear region.

INTRODUCTION

Lanekeeping assistance systems hold the promise to save thousands of lives every year by preventing unintended lane departure. Simply by keeping the vehicle in the lane, a huge number of fatal accidents can be avoided. One way to envision lanekeeping assistance is as a virtual modification to the energy of the vehicle. The system considered here virtually places the vehicle in a potential well, centered on the middle of the lane. The steering angle of the vehicle is modified to recreate the effect of driving in this virtual valley, and this gently nudges the vehicle back towards lane center. This controller is similar to a proportional plus derivative controller tracking lane center. Previous results have shown both analytically and experimentally that this lanekeeping assistance system can be guaranteed to keep the vehicle in the lane without driver input, and initial qualitative experimental results suggest it does work well with the driver.

For a system designed to keep the vehicle in the lane, a guarantee of lanekeeping performance is a necessity to ensure safety. Previous work has bounded the motion of the vehicle using a quadratic Lyapunov function. This bound ensures stability of the system, and is also used to choose controller gains for given road conditions. These bounding results rely on a linear tire model, which is valid for low levels of tire force. It is imperative that the performance of the system be guaranteed for the largest possible range of vehicle dynamics, including when the tires begin to saturate. Nonlinear tire forces could be encountered for a number of reasons, including when the speed of the vehicle is too great for a given turn, or when the driver steers inappropriately. This excessive speed could be from the driver simply going above the speed limit, or from reduced tire-road friction due to rain or ice.

One way to bound nonlinear systems is to treat the nonlinearities as a disturbance acting on a linear system. Sector bounding the nonlinearities is a classical technique that often works

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well when the nonlinearities can be written as feedback terms of system outputs. Classical solutions to the sector bounded problem [1] use the Circle Criterion or the Popov Criterion, both of which examine the transfer function of the system. In cases where the nonlinearity is well modeled however, these types of approaches usually result in a conservative bound compared to approaches that more accurately define the nonlinearity.

To remove the conservatism resulting from sector bounding, the phase plane resulting from the nonlinearities can be directly considered. For vehicles, direct analysis of the phase plane has been used successfully to analyze behavior with nonlinear tires. Inagaki et al. [2] used the sideslip vs. yaw rate and sideslip vs. sideslip-rate phase planes to explain the qualitative change in the stable region of operation during countersteering maneuvers, and used the phase plane to develop a stability controller. Using intuition from the phase plane analysis, Hoffman et al. [3] developed a criterion for stability based on the moment on the vehicle caused by the tire forces. Using this enabled investigation into the tradeoff between vehicle stability and controllability.

With the addition of a lanekeeping controller, we are now concerned with vehicle positional states as well as velocity states. With four system states, visual inspection of the phase plane is impossible. Fortunately, a numerical method called Sum of Squares programming can be used to analyze the phase plane in higher dimensions. Sum of Squares (SOS) programming is a relatively new branch of convex optimization which allows for numerical validation of the positivity of a polynomial. Jarvis-Wloszek [4] outlines many uses for SOS techniques, including nonlinear controller design and stability proofs. These SOS programs can be written as semi-definite programs, a class of problem which can be solved very efficiently.

This paper uses two distinct SOS programming methods to guarantee stability of a lanekeeping system in the presence of tire nonlinearities. The first method models the nonlinearities as sector bounded unknown disturbances, where the sector bound confines them to be above and below two linear functions of slip angle. This method finds a Lyapunov function using SOS programming that guarantees stability up to about 90% of the maximum tire force. The second method fits the nonlinear tire curve with a low order polynomial in a range of slip angles of interest, and finds a Lyapunov function valid for this polynomial system in a defined region of the phase plane. Each of these techniques finds a Lyapunov function to bound the motion of the vehicle and guarantee stability far into the nonlinear region.

VEHICLE MODEL WITH LANEKEEPING SYSTEM

The vehicle is modeled as a mass moving in the plane (Fig. 1), using the linear bicycle model. Because we are interested in lanekeeping performance, it is most useful to derive the model in coordinates representing the deviation of the vehicle from lane center. These are the lateral deviation of the center of gravity of the vehicle, e, and the angular heading error ψ representing the difference between the direction the vehicle is pointing and the direction of the road center line (see Fig. 3).

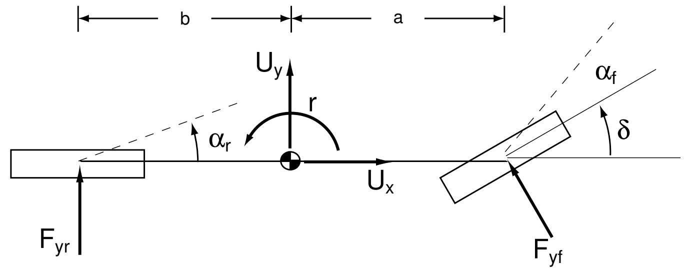

Figure 1. YAW PLANE VEHICLE MODEL

If we assume that the yaw rate is small then the motion of the left and right tires is approximately the same and we can lump the two together. This also assumes that the steering angles of the left and right front tires are equal, a valid assumption for small steering angles. We also assume that the longitudinal velocity is constant because this system is designed for highway use where the velocity is largely constant (or at most slowly varying). Thus we consider the effect of the lateral tire forces on the lateral error and heading of the vehicle.

Assuming small steering angle and heading deviation we have:

$$m\ddot{e} = F_{yf} + F_{yr} \tag{1}$$

$$I_z\ddot{\psi} = aF_{yf} - bF_{yr} \tag{2}$$

Here a and b are the distances from the center of mass of the vehicle to the front and rear axles respectively.

Vehicle Model with General Tires

The force a tire generates is a function of the slip angle at that tire. For a general nonlinear function we have:

$$m\ddot{e} = F_{yf}(\alpha_f) + F_{yr}(\alpha_r) \tag{3}$$

$$I_z\ddot{\psi} = aF_{yf}(\alpha_f) - bF_{yr}(\alpha_r) \tag{4}$$

The slip angle is defined as the angle between the direction the tire is pointing and the direction it is moving:

$$\alpha_f = atan(\frac{U_{y_f}}{U_{x_f}}) - \delta \approx \frac{U_{y_f}}{U_{x_f}} - \delta \tag{5}$$

$$\alpha_r = atan(\frac{U_{y_r}}{U_{x_r}}) \approx \frac{U_{y_r}}{U_{x_r}} \tag{6}$$

Here δ is the steering angle. This small angle approximation assumes the lateral velocities U_{y_f} and U_{y_r} are much smaller than

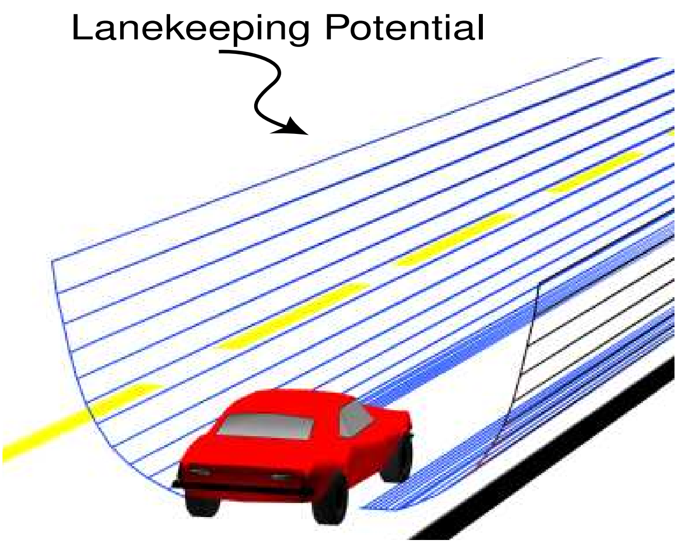

Figure 2. VISUALIZATION OF POTENTIAL FIELD

the longitudinal velocity. If we further assume the heading error and steering angle are both small, then

$$U_{y_f} \approx \dot{e} - \psi U_x + a\dot{\psi} \quad (7)$$
$$U_{y_r} \approx \dot{e} - \psi U_x - b\dot{\psi} \quad (8)$$
$$U_{x_f} \approx U_{x_r} \approx U_x \quad (9)$$

The steering angle (in the absence of driver input) is commanded by the lanekeeping controller [5]. The lanekeeping controller commands this steering angle to apply a force to the vehicle. This force is derived from an artificial potential, shown conceptually in Fig. 2, where it provides zero force on lane center and increasing force as the vehicle deviates from the specified path. The force applied to the vehicle is derived from the lateral and heading errors of the vehicle (e and ψ) as defined in Fig. 3, and is the gradient of the potential:

$$V_p(e_{la}) = k_p(e_{la})^2 = k_p(e + x_{la}\psi)^2 \quad (10)$$
$$F_{pf} = -\frac{\partial V_p}{\partial e} = -2k_p(e + x_{la}\psi) \quad (11)$$

This controller has two parameters: the potential field gain k, which represents the effective spring constant, and a lookahead distance x_{la}. This lookahead distance creates a gain on the heading error of the vehicle, and is necessary for stability at high speeds.

The steering angle from the lanekeeping controller is added to the bicycle model to obtain a model of the controlled vehicle.

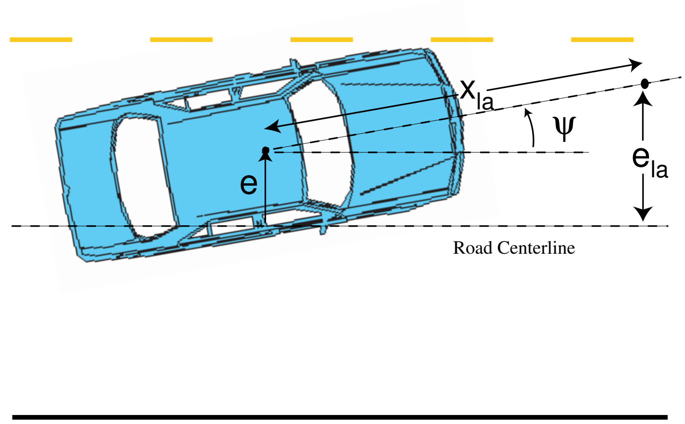

Figure 3. LATERAL AND HEADING ERRORS USED FOR CONTROLLER

The added steering angle to supply the potential field force is:

$$\delta_{pf} = -\frac{2k_p(e + x_{la}\psi)}{C_f} \quad (12)$$

With the steering command the slip angle expressions of Eqs. 5 and 6 become

$$\alpha_f = e\frac{2k_p}{C_f} + \dot{e}\frac{1}{U_x} + \psi\left(\frac{2k_p x_{la}}{C_f} - 1\right) + \dot{\psi}\frac{a}{U_x} \quad (13)$$
$$\alpha_r = \dot{e}\frac{1}{U_x} - \psi - \dot{\psi}\frac{b}{U_x} \quad (14)$$

The closed loop dynamics with the lanekeeping controller active are simply Eqs. 3 and 4 with the above expressions for the slip angles.

LINEAR VEHICLE MODEL WITH LANEKEEPING

Previous work has assumed a linear tire model. For low amounts of force, the lateral force from each set of tires is linearly proportional to the slip at those tires:

$$F_{yf} = -C_f\alpha_f \quad (15)$$
$$F_{yr} = -C_r\alpha_r \quad (16)$$

Substituting in to Eqs. 3 and 4, we have the linear bicycle model in error coordinates:

$$m\ddot{e} = -C_f\left(\frac{\dot{e}}{U_x} - \psi + \frac{a\dot{\psi}}{U_x}\right) - C_r\left(\frac{\dot{e}}{U_x} - \psi - \frac{b\dot{\psi}}{U_x}\right) + C_f\delta \quad (17)$$
$$I_z\ddot{\psi} = -aC_f\left(\frac{\dot{e}}{U_x} - \psi + \frac{a\dot{\psi}}{U_x}\right) + bC_r\left(\frac{\dot{e}}{U_x} - \psi - \frac{b\dot{\psi}}{U_x}\right) + aC_f\delta \quad (18)$$

where C_f is the front cornering stiffness. With the potential field force added in, the bicycle model in state space form is

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k_p}{m} & \frac{-(C_f+C_r)}{mU_x} & \frac{(C_f+C_r)-2k_p x_{la}}{m} & \frac{(-aC_f+bC_r)}{mU_x} \\ 0 & 0 & 0 & 1 \\ \frac{-2k_p a}{I_z} & \frac{(-aC_f+bC_r)}{I_z U_x} & \frac{(aC_f-bC_r)-2k_p x_{la} a}{I_z} & \frac{-(a^2C_f+b^2C_r)}{I_z U_x} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \psi \\ \dot{\psi} \end{bmatrix} \tag{19}$$

Previous Bounding Results

To guarantee lanekeeping performance requires a guarantee of stability. Previous bounding results [6] developed a Lyapunov function for the closed-loop system, which guarantees stability for all controller gains and vehicle speeds for which the function is valid. This Lyapunov function can then also be used to provide a numerical guarantee of lanekeeping performance when the vehicle is subjected to disturbances such as wind gusts or road curvature.

These bounding results assumed a linear tire model. This model is valid for much of highway driving, in which the tire forces are small. With a linear model, the system can be written as a Lagrangian system for certain values of the system gains. In this form, the dynamics are derived from a kinetic and a potential energy. The sum of these energies is a Lyapunov function for the system.

$$V_L = q^T P_L q \tag{20}$$

$$q = \begin{bmatrix} e \; \dot{e} \; \psi \; \dot{\psi} \end{bmatrix} \tag{21}$$

$$P_L = \begin{bmatrix} k_p & 0 & k_p a & 0 \\ 0 & .5m & 0 & 0 \\ k_p a & 0 & c_3 & 0 \\ 0 & 0 & 0 & .5I_z \end{bmatrix} \tag{22}$$

$$c_3 = k_p a(x_{la} + a) + .5(aC_f - bC_r) \tag{23}$$

VEHICLE MODEL WITH NONLINEAR TIRES

This linear tire model predicts vehicle motion quite accurately for low levels of tire force, but places no restriction on the maximum tire force attainable. Real tires do have a limit on their maximum force, equal to approximately the normal load times the coefficient of friction. A nonlinear tire model, such as the HSRI [7] model is required to capture the dynamics at large levels of force. The small angle approximations made do not need to be removed to handle higher force levels, as the maximum side force attainable is at a small slip angle (5-10 degrees for most tires).

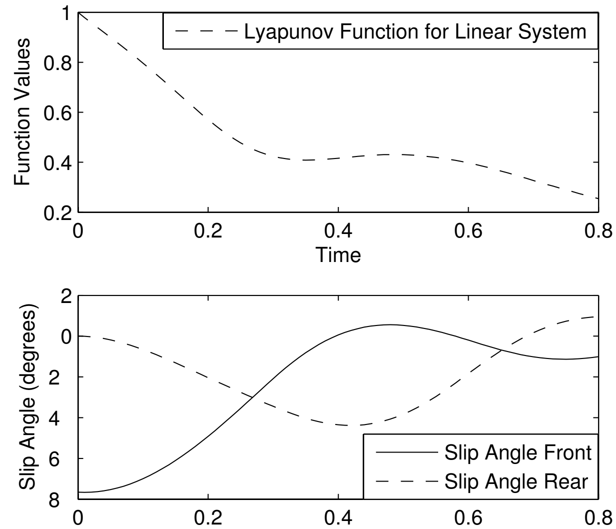

Figure 4. LYAPUNOV FUNCTION DECREASING WITH VEHICLE MOTION

The HSRI model separates tire forces into a linear region where the force is proportional to slip angle, and a nonlinear region where the force saturates. This model is given by:

$$F_{HSRI} = -C_\alpha tan(\alpha) F_H \approx -C_\alpha \alpha F_H \tag{24}$$

$$F_H = \begin{cases} 1 & H < 1/2 \\ \frac{1}{H} - \frac{1}{4H^2} & H > 1/2 \end{cases} \tag{25}$$

$$H = \left| \frac{C_\alpha tan(\alpha)}{\mu F_n} \right| \tag{26}$$

Thus for low tire forces (small slip angle, and thus small H), the tire force is proportional to slip angle. At larger slip angles the nonlinearity causes the tire force to be decreased, creating a smoothly saturating force vs. slip curve (such as the curve in Fig. 9 on page 8).

Figure 4 shows a simulation of the vehicle with lanekeeping assistance, using the HSRI tire model. The vehicle starts from an initial offset into the potential of 1 *m*, and smoothly returns to lane center. This corresponds to the vehicle starting near the edge of the lane, a large offset necessary to cause partial tire saturation. The parameters for the vehicle are shown in Table 1. The Lyapunov function initially decreases while the rear tires are in the linear region. Unfortunately when the rear tires start to saturate after about 0.4 seconds the function is no longer decreasing, and is thus not a valid Lyapunov function. This results from the fact that as the tire force decreases, less energy is dissipated. Because the Lyapunov function does not take this into account, the increase in potential causes an overall increase in the value of the

Parameter (units)	Value
U_x (m/s)	30
P.F. gain k (N/m)	7160
x_{la} (m)	14.66
m (kg)	1470
I_z (kgm^2)	2500
C_f (N/rad)	110000
C_r (N/rad)	100000
a (m)	1.0
b (m)	1.6

Table 1. CONTROLLER AND VEHICLE PARAMETERS

function.

It is clear that the Lyapunov function designed for the linear system is not valid for the vehicle with nonlinear tires. Of course it is impossible to find a global Lyapunov function for the nonlinear system, as the vehicle is not stable for extremely large slip angles. Thus we seek a function valid in some region of the state space near the origin. The need to find a Lyapunov function valid in a specific region, and the complicated nonlinear dynamics together make it infeasible to find a Lyapunov function by inspection.

THE POSITIVSTELLENSATZ

To develop a bound applicable into the nonlinear region of tire operation, we use a theorem from algebra called the Positivstellensatz. This theorem establishes conditions to prove that a set is empty. While the general theorem (see [8], [4]) is beyond the scope of this paper, the specific form of it used here can be written as follows:

$$\begin{gathered}\{x \in R^n | f_1(x) \geq 0, f_2(x) \geq 0, g \neq 0\} = \emptyset \\ \Updownarrow \\ \exists\, s_1 \dots s_4, k \;\texttt{s.t.}\; s_1 + s_2 f_1 + s_3 f_2 + s_4 f_1 f_2 + g^k = 0\end{gathered} \tag{27}$$

where s_i is a sum of squares polynomial, of the form:

$$s(x) = \sum_{i=1}^{s} p_i^2(x) \tag{28}$$

where each p_i is a polynomial.

In words, this says that we have two equivalent conditions: If we can find the sum of squares (SOS) polynomials $s_1 \dots s_4$ then we know that there are no values of x that violate the inequalities and the inequation. The theorem also goes the other direction, saying that if there are no such values of x, then there do exist polynomials to make the second statement hold.

This is a useful result for this work because it can be used to show that a Lyapunov function is decreasing within a certain region. First however, we need to be able to find these polynomials in an efficient manner.

Sum Of Squares Programs

By reorganizing Eq. 27 we can write it as a Sum of Squares Program:

$$-s_2 f_1 - s_3 f_2 - s_4 f_1 f_2 - g^k = s_1 \tag{29}$$

Here we seek coefficients for s_2, s_3, s_4 such that the whole expression on the left hand side is a sum of squares.

This problem can be turned into a Semi-Definite Program and solved very efficiently. Do this, we write the polynomial in matrix form:

$$\begin{gathered}a + bx_1 + cx_2 + dx_1x_2 + ex_1^2 + fx_2^2 + \dots = \\ \begin{bmatrix} 1 & x_1 & x_2 & \dots \end{bmatrix} \begin{bmatrix} a & .5b & .5c & \\ .5b & e & .5d & \dots \\ .5c & .5d & f & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}\end{gathered} \tag{30}$$

For a general problem of the form of Eq. 29, some coefficients in this matrix would be known and others unknown. We would also have constraints relating some of the coefficients to others. This is simply a Semi-Definite Program [8].

Lyapunov Function Constraints as SOS program

We can phrase the general Lyapunov function requirements in the form of the Positivstellensatz as follows:

$$\begin{gathered}\{x \in R^n | \dot{V}(x) \geq 0, g(x) \geq 0, ||x||^2 \neq 0\} = \emptyset \\ \Updownarrow \\ \exists\, s_0 \dots s_4, k \;\texttt{s.t.}\; -s_2 g - s_3 \dot{V} - s_4 g \dot{V} - (x^T x)^k = s_1\end{gathered} \tag{31}$$

Here $g(x)$ is a function used to constrain the region over which the Lyapunov function must be decreasing. Thus we are requiring that the Lyapunov function be decreasing whenever $g(x)$ is positive.

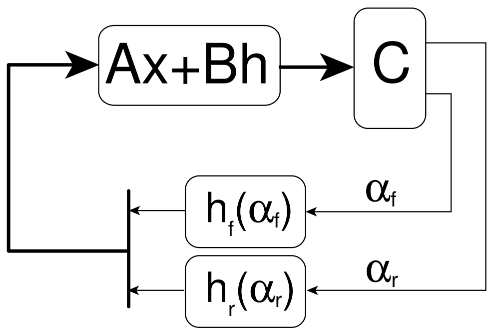

Figure 5. LUR'E SYSTEM BLOCK DIAGRAM

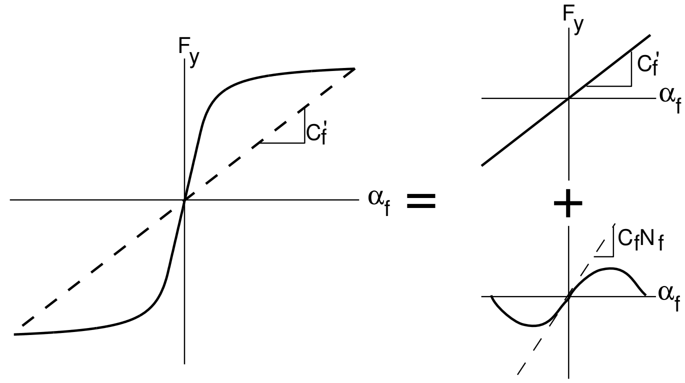

Figure 6. TIRE FORCE DECOMPOSITION

TIRE NONLINEARITY MODELED AS SECTOR BOUNDED NONLINEARITY

One classical way to represent a nonlinearity is with a sector bound [1]. Known as a Lur'e problem, the system must be decomposed into a linear system with a sector bounded nonlinear feedback. This method was used classically with analytical transfer function techniques, but can now be used much more effectively with the sum of squares programming outlined in the previous section.

Figure 5 shows the linear and nonlinear components of this model. In the case of the vehicle, the nonlinearity is a function of the wheel slip angles, so these must be the outputs of the system. Here $Ax + Bh$ is the linear plant, with outputs α_{front} and α_{rear}. The only input is the force resulting from the tire nonlinearity. The driver command is assumed to be zero in this model. In state space form, this model is simply:

$$\dot{x} = [A]\,\dot{x} + [B]\,h(y) \qquad\qquad y = Cx \tag{32}$$

where $x = \begin{bmatrix} e\ \dot{e}\ \psi\ \dot{\psi} \end{bmatrix}^T$ To complete this model, we need to bound the nonlinear function $h(y)$, and develop the A,B and C matrices for the linear portion.

We need to decompose the tire force into a linear portion and a nonlinear portion as shown in Fig. 6. The nonlinear portion is constrained to be a fraction of the maximum force, N_f and N_r for the front and rear respectively. Thus we are bounding the nonlinearity such that $h_f < C_f N_f \alpha_f$ and $h_r < C_r N_r \alpha_r$, where $0 < N_f < 1$, $0 < N_r < 1$. The rest of the tire force is then proportional to slip angle, so we can express the total force as the sum of a linear term and the nonlinearity:

$$F_{front} = -C_f(1-N_f)\alpha_f + h_f(\alpha_f) = -C_f{}'\alpha_f + h_f(\alpha_f) \tag{33}$$
$$F_{rear} = -C_r(1-N_r)\alpha_r + h_r(\alpha_r) = -C_r{}'\alpha_r + h_r(\alpha_r) \tag{34}$$

Here $C_f{}'$ and $C_r{}'$ are the proportionality constant relating force to slip for the linear portion of the system and are distinct from (and smaller than) the actual cornering stiffnesses of the vehicle.

This decomposition of the tire force can be expressed in terms of the HSRI tire model outlined earlier. From the HSRI model we have:

$$F_{HSRI} = -C_\alpha F_H \alpha \tag{35}$$
$$= -C_\alpha{}'\alpha - \alpha(C_\alpha F_H - C\alpha') \tag{36}$$

Thus the nonlinear function $h = -\alpha(C_\alpha F_H - C_\alpha{}')$.

To fit the form of equation 31, a function is needed that is negative when the force is within the sector.

$$g_1 = (h_f(\alpha_f) - C_f N_f \alpha_f) h_f(\alpha_f) \leq 0 \tag{37}$$
$$g_2 = (h_r(\alpha_r) - C_r N_r \alpha_r) h_r(\alpha_r) \leq 0 \tag{38}$$

Figure 6 shows the actual force vs. slip relationship based on the HSRI model, but all the sector bound requires is that the nonlinear portion be within the sector defined by the dashed line.

The A matrix must then represent the portion of the tire force that is proportional to slip angle

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2kC_f{}'}{mC_f} & \frac{-(C_f{}'+C_r{}')}{mU_x} & \frac{(C_f{}'+C_r{}')-2kx_{la}\frac{C_f{}'}{C_f}}{m} & \frac{(-aC_f{}'+bC_r{}')}{mU_x} \\ 0 & 0 & 0 & 1 \\ \frac{-2kaC_f{}'}{I_zC_f} & \frac{(-aC_f{}'+bC_r{}')}{I_zU_x} & \frac{(aC_f{}'-bC_r{}')-2akx_{la}\frac{C_f{}'}{C_f}}{I_z} & \frac{-(a^2C_f{}'+b^2C_r{}')}{I_zU_x} \end{bmatrix} \tag{39}$$

Basically this system matrix is the normal vehicle system, but with the reduced cornering stiffness representing the linear portion of the vehicle. The other difference from the standard model (Eqn. 19) is that the terms involving the steering from the lanekeeping controller are multiplied by $\frac{C_f{}'}{C_f}$. This represents the fact that the lanekeeping controller steering is based on the cornering stiffness C_f, but the actual force is based on the reduced cornering stiffness $C_f{}'$.

The C matrix is formed to make the front and rear slip angles be the outputs so that the nonlinearities can act on the slip angles in feedback.

$$C = \begin{bmatrix} \frac{k}{C_f} & \frac{1}{U_x} & -1+\frac{kx_{la}}{C_f} & \frac{a}{U_x} \\ 0 & \frac{1}{U_x} & -1 & \frac{-b}{U_x} \end{bmatrix} \quad (40)$$

The B matrix represents the way in which the forces from the nonlinearities enter the system.

$$B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m} & \frac{1}{m} \\ 0 & 0 \\ \frac{a}{m} & -\frac{b}{m} \end{bmatrix} \quad (41)$$

For a quadratic Lyapunov function V as in Eq. 20, we want the derivative of the Lyapunov function to be negative whenever the nonlinearity is within the sector. This can be expressed in matrix form [9]:

$$g_1 \le 0, g_2 \le 0 \Rightarrow \dot{V} = \begin{bmatrix} x^T & h^T \end{bmatrix} \begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \le 0 \quad (42)$$

Now we can express the requirements on the Lyapunov function in a form to fit the Positivstellensatz. Specifically, we want V to be positive, and $\dot{V}$ to be negative whenever h is in the sector (g is negative). Thus we seek polynomials such that,

$$P > 0 \quad (43)$$

$$s_3g_1 + s_4g_2 - \dot{V} = s_1 \quad (44)$$

This is the standard form of the Positivstellensatz, except that $\dot{V}$ is not multiplied by an SOS polynomial. This is required to keep the problem linear in the unknown polynomial coefficients. The coefficients of V are unknowns, so they cannot be multiplied by another unknown.

This problem can be readily solved with the SOSTools, a Sum of Squares toolbox for Matlab [10]. The numerical Lyapunov function found for this example is:

$$P_{sec} = \begin{bmatrix} 1.00 & .097 & .383 & .007 \\ .097 & .149 & .012 & .004 \\ .382 & .012 & 5.10 & .120 \\ .007 & .004 & .120 & .064 \end{bmatrix} \quad (45)$$

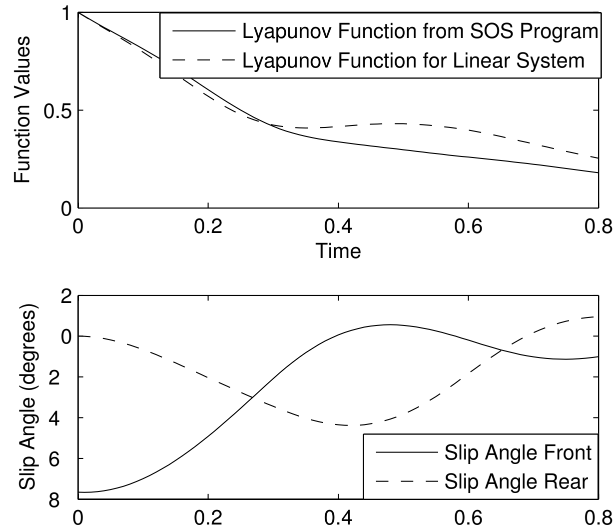

Figure 7. LYAPUNOV FUNCTION VALUES AND VEHICLE STATES

Figure 7 shows the Lyapunov function value for an initial 1m offset from lane center. The Lyapunov function for the linear system is shown for reference, as it increases at about .4s when the rear slip angle becomes large. The Lyapunov function found for the sector bounded system decreases through this section. The function found has far more nonzero terms than the function for the linear system (Eq. 20), and the total combination of these terms results in a function that decreases even as the tires start to saturate.

Figure 8 shows the region of the tire curve where the Lyapunov function is valid. The function is valid up to about 90% of the maximum tire force. This number does depend on the exact lanekeeping and vehicle parameters, suggesting that these can be chosen to maximize stability of the vehicle at the limits of handling.

This function guarantees that the system with lanekeeping active is stable up to 90% of the maximum tire force. This can be used to guarantee that if the system starts with the tires not saturated, they will not fully saturate. Because the value of the Lyapunov function value can not increase with time, the maximum lane deviation can be calculated for a given initial condition by finding the maximum deviation with energy equal to the initial value. As in previous work with the linear system, this Lyapunov function can also be used to prove that the vehicle will stay in the lane when subjected to disturbances such as wind or road curvature. Further details of this approach are detailed in [6].

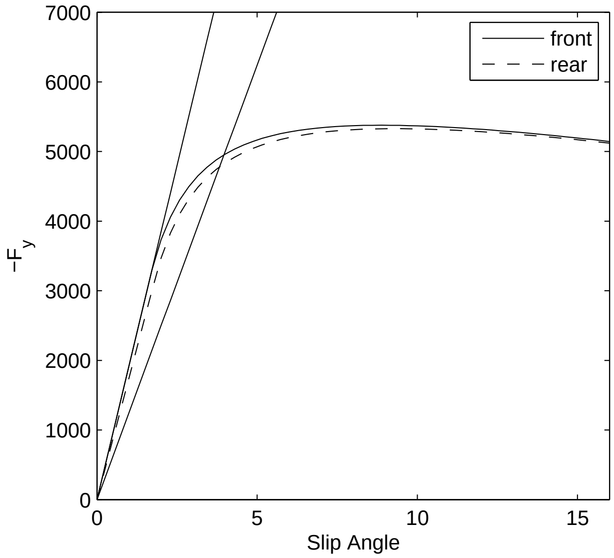

Figure 8. REGION OF VALIDITY FOR SECTOR BOUNDED NONLINEARITY LYAPUNOV FUNCTION

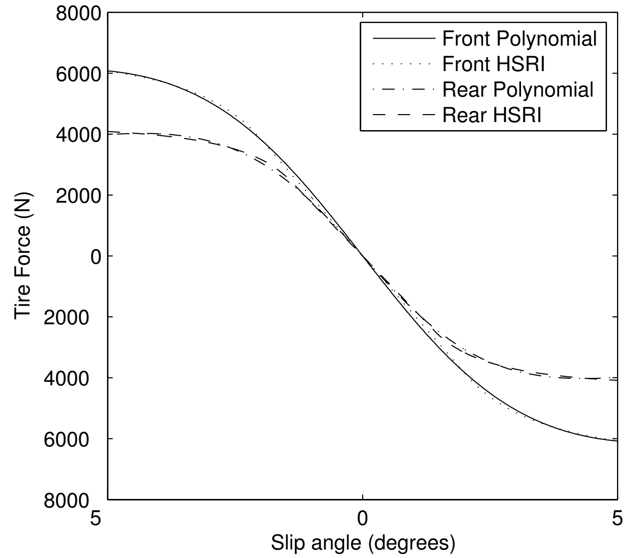

Figure 9. TIRE MODEL POLYNOMIAL FIT

TIRE NONLINEARITY MODELED WITH POLYNOMIAL DYNAMICS

To bound the dynamics further into the nonlinear region of the state space, a more accurate model is needed. To use the Positivstellensatz in the form of Eq. 27, the dynamics need to be in the form of polynomials. Here we fit a rational polynomial to the HSRI tire model [7].

Using least squares we fit to the following polynomial:

$$f(\alpha) = \frac{c_n\alpha^n + c_{n-1}\alpha^{n-1} + \ldots + c_1\alpha}{d_m\alpha^m + d_{m-1}\alpha^{m-1} + \ldots + d_1\alpha + 1} \qquad (46)$$

Where n and m are the orders of the polynomials for the numerator and denominator. A fit for each tire with orders of 1 and 2 is shown in Fig. 9. The fit is quite close, even with the low order polynomials used, which is not surprising given the existence of low order polynomial tire models in the literature (for example, see [11]). These models could also be considered in place of the curve fit used here.

Now that we have a tire model in polynomial form, we can plug this into Eqs. 3 and 4 to find a polynomial model for the vehicle.

$$\ddot{e} = \frac{N_e(e, \dot{e}, \psi, \dot{\psi})}{D(e, \dot{e}, \psi, \dot{\psi})} \qquad \ddot{\psi} = \frac{N_\psi(e, \dot{e}, \psi, \dot{\psi})}{D(e, \dot{e}, \psi, \dot{\psi})} \qquad (47)$$

This model still assumes small angles, but the force from the tires is now based on the polynomial fit of the HSRI model. As in the sector bounded case, this is done because the slip angle at which the tire force starts to saturate is only about 2 degrees, so small angle approximations are very accurate. The denominators of the two expressions in the above equation are the same because the tire model enters in a similar manner in the two expressions of Eqs. 3 and 4.

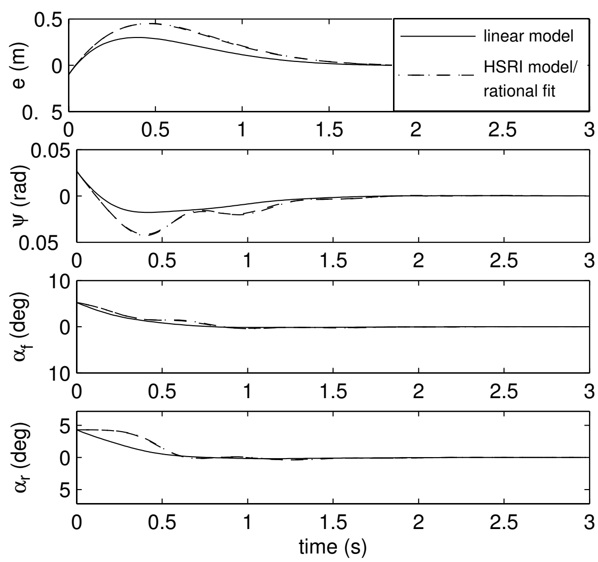

Figure 10. DYNAMIC COMPARISON OF CURVE FIT TO HSRI MODEL

To confidently use this polynomial model to bound the vehicle motion it needs to be verified dynamically. Figure 10 shows the response of the system with the vehicle initially to the right of the centerline, but travelling and pointing to the left ($x_0 = \begin{bmatrix} -0.1m & 2.7m/s & 1.5deg & -14deg/s \end{bmatrix}^T$). Because the slip angles are well into the nonlinear region of the tires, the linear model differs from the HSRI model. The polynomial model tracks the HSRI model extremely well, with the lines directly on top of each other on the plot. This curve fit is only accurate up to the peak force attainable by the tires (because the fit was only performed up to that point), but it fits extremely well to that point. This is sufficient, because we are interested in bounding the motion of the vehicle below the peak force.

We now use the result from the Positivstellensatz, Eq. 31, with these polynomial dynamics plugged in. We use the function g to constrain the solution to a region of the state space. Clearly we will not be able to find a Lyapunov function for the entire state space because the system is not stable for tire forces beyond the peak force. This function g could be any polynomial. Here, for simplicity, an ellipse is used:

$$g_{sect} = 1 - 6.25e^2 - 0.411\dot{e}^2 - 319\psi^2 - 0.925\dot{\psi}^2. \qquad (48)$$

This function is chosen to match the slip angle range for which the tire curve fit is valid.

The derivative of the Lyapunov function is found by plugging in these polynomial dynamics:

$$\dot{V} = \frac{\partial V}{\partial e}\dot{e} + \frac{\partial V}{\partial \dot{e}}\ddot{e} + \frac{\partial V}{\partial \psi}\dot{\psi} + \frac{\partial V}{\partial \dot{\psi}}\ddot{\psi} \qquad (49)$$

$$\dot{V} = \frac{\partial V}{\partial e}\dot{e} + \frac{\partial V}{\partial \dot{e}}\frac{N_e}{d} + \frac{\partial V}{\partial \psi}D + \frac{\partial V}{\partial \dot{\psi}}\frac{n_\psi}{D} \qquad (50)$$

Because the denominator of the dynamics (and hence of $\dot{V}$) is always positive, we can multiply through by this term.

$$\dot{V}D = \frac{\partial V}{\partial e}\dot{e}D + \frac{\partial V}{\partial \dot{e}}N_e + \frac{\partial V}{\partial \psi}\dot{\psi}D + \frac{\partial V}{\partial \dot{\psi}}N_e \qquad (51)$$

Where D, N_e and N_ψ are from Eq. 47. Because D is always positive, requiring $\dot{V}$ to be negative is equivalent to requiring $\dot{V}D$ to be negative. Plugging this into Eq. 31 yields the Positivstellensatz condition for the Lyapunov function. Here we are seeking the polynomials $s_1 \dots s_4$ and also the Lyapunov function itself. To keep the problem linear, we must preselect the SOS polynomials s_4 and s_3. Here they are chosen to be the constant value 1. Because of the requirement to make this choice there will not always exist a SOS decomposition even when the Lyapunov function is decreasing in the specified region.

For this polynomial model, the approach does find a Lyapunov function decreasing in the defined region:

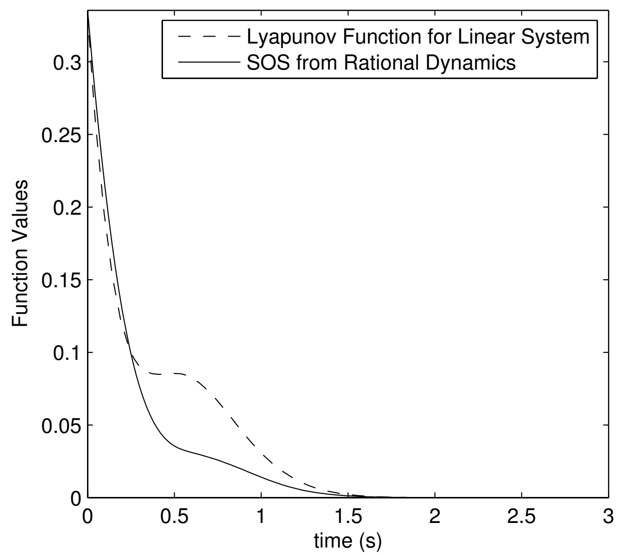

Figure 11. LYAPUNOV FUNCTION FROM RATIONAL DYNAMICS

$$P_{rat} = \begin{bmatrix} 1.87 & .31 & .12 & .05 \\ .31 & .48 & 0 & .10 \\ .12 & 0 & 7.29 & .13 \\ .05 & .10 & .13 & .11 \end{bmatrix} \qquad (52)$$

Similar to the sector bounded case, this function has more nonzero terms than the linear Lyapunov function. Figure 11 shows the value of this function for the trajectory of Fig. 10, and it decreases as predicted. This Lyapunov function is valid for all slip angles which are in the region of interest (defined by the function g_{sect}) and for which the tire curve fit is accurate.

CONCLUSIONS AND FUTURE WORK

The techniques described here are a step towards bounding the motion of lanekeeping assisted vehicles with tire saturation. The two methods each are able to guarantee stability quite far into the nonlinear region of tire forces. This means that stability can be assured even when the tires have begun to saturate.

The work described here is to find the Lyapunov function, thus guaranteeing stability. The next step is to use the Lyapunov function to provide a numerical bound on the motion of the vehicle. Performance of the Lyapunov function as a bound, and how this varies with vehicle and controller parameters, is key to the utility of this method. Extending this bound to time varying disturbances would be useful to guarantee performance in the presence of wind, road bank, or other slowly varying disturbances.

The polynomial fit to the tire curve can also be extended, and possibly combined with the sector bounding. By considering tire forces between two polynomial bounds, uncertainty in road surface conditions or tire properties could be directly considered. This tighter sector bound should result in a less conservative approach, as it more accurately models the system.

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